Subject: Mathematics

Paper-I, Unit – I: Matrices, Semester-II

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Example-1: Let A be a square matrix of order n. Then prove that

(i) $A+A'$ is symmetric

(ii) $A-A'$ is skew-symmetric

(iii) AA', $A'A$ are symmetric.

Solution-(i) As $(A+A')'=(A)'+(A')'=A'+A=A+A'$. So, $A+A'$ is symmetric.

(ii) As $(A-A')'=(A)'-(A')'=A'-A= --(A-A')$. So, $A-A'$ is skew-symmetric.

(iii) As $(AA')'=(A')'(A)'=AA'$ and $(A'A)'=(A)'(A')'=A'A$. So, AA', $A'A$ are symmetric.

Example-2: Let A and B are symmetric/ skew-symmetric matrices of same order. Then prove that $aA+bB$ and $aA-bB$ are also symmetric/ skew-symmetric matrices for some scalers a and b.

Solution- As A and B are symmetric matrices then $A'=A$, $B'=B$.

Now $(a A +b B)'= (a A)'+ (b B)'= a(A)'+b(B)'=aA+bB$ and

$(a A -b B)'= (a A)'- (b B)'= a(A)'-b(B)'=aA-bB$. So $aA+bB$ and $aA-bB$ are also symmetric matrices.

Similarly, we can show that if A and B are skew-symmetric matrices then $aA+bB$ and $aA-bB$ are also skew-symmetric matrices.

Example-3: Let A and B are symmetric matrices of same order. Then AB is also symmetric if and only if AB = BA.

Solution- As A and B are symmetric matrices then $A'=A$, $B'=B$.

Now AB is symmetric $\Leftrightarrow (AB)'=(AB)$

$\Leftrightarrow B'A'=AB$

$\Leftrightarrow BA=AB.$

Example-4: Prove that

(i) The adjoint of a symmetric matrix is also symmetric.

(ii) The inverse of a non-singular symmetric matrix is symmetric.

(iii) If A^2 is symmetric then either A is symmetric or skew-symmetric.

Solution- (i) Let A be a symmetric matrix then A'=A.

Since (adj A)' = adj (A)' ⇒ (adj A)' = adj A. So adj A is also symmetric.

(ii) Let A be a non-symmetric matrix then A'=A and let A^{-1} is the inverse of A.

As $(A^{-1})' = (A')^{-1} \Rightarrow (A^{-1})' \Rightarrow A^{-1}$. So A^{-1} is also symmetric.

(iii) Let A be a symmetric matrix then A'=A.

Then $(A^2)' = (AA)' = A'A' = AA = A^2$.

Let A be a skew-symmetric matrix then A'=-A.

Then $(A^2)' = (AA)' = A'A' = (-A)(-A) = A^2$.

Hence A^2 is symmetric whether A is symmetric or skew-symmetric.

Example-5: Express the following matrix as the sum of a symmetric and a skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

Solution- For any matrix A, we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C$$

Where $B = \frac{1}{2}(A + A')$ is symmetric and $C = \frac{1}{2}(A - A')$ is skew symmetric matrices.

Now

$$B = \frac{1}{2}(A + A') = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 0 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A') = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}.$$

Example-6: Prove that the determinant of an orthogonal matrix is either +1 or -1.

Solution- Let A be an orthogonal matrix then AA’=I.

Now, $|AA'| = |I| \Rightarrow |A|.|A'| = 1 \Rightarrow |A|.|A| = 1 \Rightarrow |A|^2 = 1.(as\ |A'| = |A|)$

$\Rightarrow |A| = \pm 1.$

Example-7: Let A and B are orthogonal matrices then prove that AB, BA are also orthogonal.

Solution- Since A and B are orthogonal matrix then AA’=I =A’A and BB’=I =B’B.

Now $(AB).(AB)' = (AB)(B'A') = A(BB')A' = A(I)A' = AA' = I.$

Similarly, $(BA).(BA)' = I$. Hence AB, BA are also orthogonal.

Example-8: Prove that the inverse of an orthogonal matrix is orthogonal.

Solution- Let A is orthogonal matrix then AA’=I =A’A.

Now $(AA')^{-1} = I^{-1} = (A'A)^{-1} \Longrightarrow (A')^{-1}(A)^{-1} = I = (A)^{-1}(A')^{-1}$

$\Longrightarrow (A^{-1})'(A)^{-1} = I = (A)^{-1}(A^{-1})' \ \ \{as\ (A')^{-1} = (A^{-1})'\}$

Hence $(A)^{-1}$ is also orthogonal.

Example-9: Prove that the following matrix is orthogonal:

$$A = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}.$$

Solution- Since $A = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \Longrightarrow A' = \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$

Now $AA' = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, given matrix is orthogonal.

Example-10: Prove that the modulus of the determinant of a unitary matrix is unity.

Solution- Let A be a unitary matrix then AA*=I=A*A.

Now, $|AA^*| = |I| \Rightarrow |A|.\left|\overline{(A)}'\right| = 1 \Rightarrow |A|.|\bar{A}| = 1 \Rightarrow |A|.\overline{|A|} = 1$

$\Rightarrow |A|^2 = 1.(as\ |A'| = |A|\ , |\bar{A}| = \overline{|A|}\ and\ z\bar{z} = (modulus\ of\ z)^2\ )$

$\Rightarrow |A| = \pm 1.$

Example-11: Let A and B are unitary matrices then prove that AB, BA are also unitary.

Solution- Since A and B are unitary matrix then AA*=I =A*A and BB*=I =B*B.

Now $(AB).(AB)^* = (AB)(B^*A^*) = A(BB^*)A^* = A(I)A^* = AA^* = I.$

Similarly, $(BA).(BA)^* = I$. Hence AB, BA are also unitary.

Example-12: Prove that the inverse of a unitary matrix is unitary.

Solution- Let A is a unitary matrix then AA*=I =A*A.

Now $(AA^*)^{-1} = I^{-1} = (A^*A)^{-1} \Longrightarrow (A^*)^{-1}(A)^{-1} = I = (A)^{-1}(A^*)^{-1}$

$\Longrightarrow (A^{-1})^*(A)^{-1} = I = (A)^{-1}(A^{-1})^*\ \{as\ (A^*)^{-1} = (A^{-1})^*\}$

Hence $(A)^{-1}$ is also unitary.

Example-13: Prove that the following matrix is unitary:

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

Solution- Since $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \Longrightarrow A^* = \overline{(A)}' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

Now $AA^* = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}.\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, given matrix is unitary.

Example-14: Let A be a square matrix of order n. Then prove that

(i) A+A* is Hermitian

(ii) A-A* is skew-Hermitian

(iii) AA*, A*A are Hermitian.

Solution-(i) As (A+A*) *=(A)*+(A*) *=A*+A=A+A*. So, A+A* is Hermitian.

(ii) As (A-A*) * = (A)*-(A*) *=A*-A= --(A-A*). So, A-A* is skew-Hermitian.

(iii) As (AA*) *=(A*) *(A)*=AA* and (A*A) *=(A)*(A*) *=A*A. So, AA*, A*A are Hermitian.

Example-15: Let A and B are Hermitian/ skew-Hermitian matrices of same order. Then prove that aA+bB and aA-bB are also Hermitian/ skew-Hermitian matrices for some scalers a and b.

Solution- As A and B are Hermitian matrices then A*=A, B*=B.

Now (a A +b B) *= (a A) *+ (b B) *= a(A)*+b(B)*=aA+bB and

(a A -b B) *= (a A) *- (b B) *= a(A)*-b(B)*=aA-bB. So aA+bB and aA-bB are also Hermitian matrices.

Similarly, we can show that if A and B are skew-Hermitian matrices then aA+bB and aA-bB are also skew-Hermitian matrices.

Example-16: Let A and B are Hermitian matrices of same order. Then AB is also Hermitian if and only if AB = BA.

Solution- As A and B are Hermitian matrices then A*=A, B*=B.

Now AB is Hermitian ⟺ (AB)*=(AB)

⟺ B*A*=AB

⟺ BA=AB.

Example-17: Express the following matrix as the sum of a Hermitian and a skew-Hermitian matrix:

$$A = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$$

Solution- For any matrix A, we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = B + C$$

Where $B = \frac{1}{2}(A + A^*)$ is Hermitian and $C = \frac{1}{2}(A - A^*)$ is skew Hermitian matrices.

Now

$$A=\begin{bmatrix}-2+3i & 1-i & 2+i\\ 3 & 4-5i & 5\\ 1 & 1+i & -2+2i\end{bmatrix} \Longrightarrow A^{*}=\begin{bmatrix}-2-3i & 3 & 1\\ 1+i & 4+5i & 1-i\\ 2-i & 5 & -2-2i\end{bmatrix}$$

$So,\qquad B=\frac{1}{2}(A+A^{*})$

$$=\frac{1}{2}\left\{\begin{bmatrix}-2+3i & 1-i & 2+i\\ 3 & 4-5i & 5\\ 1 & 1+i & -2+2i\end{bmatrix}+\begin{bmatrix}-2-3i & 3 & 1\\ 1+i & 4+5i & 1-i\\ 2-i & 5 & -2-2i\end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix}-4 & 4-i & 3+i\\ 4+i & 8 & 6-i\\ 3-i & 6+i & -4\end{bmatrix}$$

$and\qquad C=\frac{1}{2}(A-A^{*})$

$$=\frac{1}{2}\left\{\begin{bmatrix}-2+3i & 1-i & 2+i\\ 3 & 4-5i & 5\\ 1 & 1+i & -2+2i\end{bmatrix}-\begin{bmatrix}-2-3i & 3 & 1\\ 1+i & 4+5i & 1-i\\ 2-i & 5 & -2-2i\end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix}6i & -2-i & 1+i\\ 2-i & -10i & 4+i\\ -1+i & -4+i & 4i\end{bmatrix}$$

So,

$$A=\frac{1}{2}\begin{bmatrix}-4 & 4-i & 3+i\\ 4+i & 8 & 6-i\\ 3-i & 6+i & -4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}6i & -2-i & 1+i\\ 2-i & -10i & 4+i\\ -1+i & -4+i & 4i\end{bmatrix}.$$