Subject: Mathematics

Paper-I, Unit – I: Matrices, Semester-II

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Matrices: A rectangular arrangement of mn numbers or quantities in m rows and n columns is called a matrix of order $m \times n$. It is denoted as $A = [a_{ij}]_{m \times n}$. The elements a_{ii} are called diagonal elements of the matrix.

Types of matrices:

(1) **Square Matrix**-If a matrix has equal number of rows and columns, it is called a square matrix. Its order will be $m \times m$ or $n \times n$ or n^2, e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2x2}$$

(2) **Row Matrix**-. If a matrix has only one row and any number of columns, it is called a Row matrix. Its order will be $m \times 1$, e.g.,

$$[1\ 3\ 6\ 9]_{1x4}$$

(3) **Column Matrix** - A matrix, having one column and any number of rows, is called a Column matrix. Its order will be $1 \times n$,e.g., $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}_{3x1}$

(4) **Triangular Matrix**- A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix*. A square matrix, all of whose elements above the leading diagonal are zero, is called *a lower triangular matrix* e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

(5) **Diagonal Matrix**. A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(6) **Unit or Identity Matrix**. A square matrix is called an identity or unit matrix if all the diagonal elements are unity and non-diagonal elements are zero. An identity matrix of order $n \times n$ is denoted as $I_{n \times n}$,e.g.,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(7) **<u>Null Matrix (Zero Matrix</u>**)- Any matrix, in which all the elements are zeros, is called a Zero matrix or Null matrix, e.g.,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(8) **<u>Scalar Matrix</u>**. A diagonal matrix in which all the diagonal elements are equal to a scalar, say (c) is called a scalar matrix.

For example;

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

i.e., $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $[a_{ij}] = \begin{cases} 0, & when\ i \neq j \\ c, & when\ i = j \end{cases}$

(9) **<u>Transpose of a Matrix</u>**. On interchange of the rows and the corresponding columns in a given matrix *A*, the new matrix obtained is called the transpose of the matrix *A* and is denoted by A' or A^T e.g.,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

Also (i) $(A')' = A$ (ii) $(A+B)' = A' + B'$ (iii) $(AB)' = B'A'$ (Reversal Law)

(10) **<u>Symmetric Matrix</u>**. A square matrix will be called symmetric, if for all values of i and j, $a_{ij} = a_{ji}$ i.e., $A' = A$

e.g., $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(11) **<u>Skew Symmetric Matrix</u>**. A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of i and j or $A' = -A$

(2) all diagonal elements are zero, e.g.,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(l2) **<u>Conjugate Transpose of Matrix</u>**. Transpose of the conjugate (or vice-versa) of a matrix A is denoted by A^{θ} or A^*. So $A^* = (\bar{A})'$ or $\overline{(A')}$

Let $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 1+i & 2+3i & 4 \\ 7+2i & +i & 3+2i \end{bmatrix}$$

$$(\overline{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

Also (i) $(A^*)^* = A$ (ii) $(A+B)^* = A^* + B^*$ (iii) $(AB)^* = B^* A^*$ (Reversal Law)

(13) **<u>Hermitian Matrix</u>**. A square matrix $A = (a_{ij})$ will be called a Hermitian matrix if every i- jth element of A is equal to conjugate complex j-ith element of A.

In other words, $a_{ij} = \overline{a_{ji}}$

All the elements in the principal diagonal will be of the form

$$a_{ii} = \overline{a}_{ii} \qquad \text{or} \qquad a_{ii} - \overline{a}_{ii} = 0$$

if $a_{ii} = a + ib$ then $\overline{a}_{ii} = a - ib$

$$(a+ib) - (a-ib) = 0 \quad \Rightarrow \quad 2ib = 0 \Rightarrow b = 0$$

So, $a_{ii} = a + 0 = a.$. Also if $a = 0 \Rightarrow a_{ii} = 0.$

Hence, all the diagonal elements of a Hermitian Matrix are always real.

e.g. $$\begin{bmatrix} 1 & 2-3i & 4+5i \\ 2+3i & 0 & 2i \\ 4-5i & -2i & -3 \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that

$$A^{\theta} = A$$

$$(\overline{A})' = A$$

(14) **<u>Skew Hermitian Matrix</u>**. A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every i-jth element of A is equal to negative conjugate complex j-ith element of A.

In other words, $a_{ij} = -\overline{a}_{ji}$

All the elements in the principal diagonal will be of the form

$$a_{ii} = -\overline{a}_{ii} \qquad \text{or} \qquad a_{ii} = -\overline{a}_{ii} = 0$$

if $a_{ii} = a + ib$ then $\overline{a}_{ii} = a - ib$

$$(a+ib) + (a-ib) = 0 \quad \Rightarrow \quad 2a = 0 \Rightarrow a = 0$$

$\Rightarrow a_{ii} = 0 + ib = ib$ .So a_{ii} is pure imaginary or 0.

Hence, all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g. $$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that

$$A^{\theta} = -A$$

$$(\overline{A})' = -A$$

(15) **Idempotent Matrix**. A square matrix, such that $A^2 = A$ is called Idempotent Matrix.

e.g. $$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix},\ A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

(16) **Periodic Matrix**. A square matrix A will be called a Periodic Matrix, if

$$A^{k+1} = A$$

Where k is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. if we choose $k = 1$, we get $A^2 = A$ and we call it to be idempotent matrix.

(17) **Nilpotent matrix**. A square matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where k is a +ve integer; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

e.g., $$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

A is nilpotent matrix whose index is 2.

(18) **Involuntary Matrix**. A square matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always $\therefore$ Unit matrix is involuntary.

(19) **Equal Matrices**. Two matrices are said to be equal if

(i) They are of the same order.

(ii) The elements in the corresponding positions are equal.

Thus if $$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Here $$A = B$$

(20) **Singular Matrix**. If the determinate of the square matrix is zero, then the matrix is known as singular matrix e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular matrix, because $|A| = 6 - 6 = 0$.

(21) **Orthogonal Matrix**. A square matrix 'A' is called orthogonal if

$$AA' = A'A = I(identity\ matrix).$$

(22) **Unitary Matrix.** A square matrix 'A' is called unitary if

$$AA' = A'A = I(identity\ matrix).$$

Rank of Matrix (Echelon & Normal Form)

Rank of a Matrix- The number of non-zero rows in a matrix when it is either in Echelon or Normal form is called rank of the matrix. The rank of matrix A is denoted as rank(A) or $\rho(A)$.

The rank of a matrix is said to be r if

(a) It has at least one non-zero minor of order r.

(b) Every minor of A of order higher than r is zero.

Note: (i) Non-zero row is that row in which all the elements are not zero.

(ii) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

(iii) Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Normal Form (Canonical Form)

By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A:

(i) I (ii) $[I_r \quad 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A. Since both row and column transformations may be used here, the element 1 of the first row obtained can be moved in the first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

Example- Find the rank of the matrix.

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$$

Solution- Here, we have

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 11 & 19 & 26 \end{bmatrix} (R_2 \to R_2 + 3R_1)$$

The number on non-zero row is 2, therefore Rank $(A) = 2$

Rank of matrix by triangular form(Echelon Form)

Rank = Number of non-zero row in upper or lower triangular matrix.

Note. Non-zero row is that row which does not contain all the elements as zero.
Example-Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & I \\ I & 3 & 4 & 5 \end{bmatrix}$$

Solution-
$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & I \\ I & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Rank = Number of non zero rows $= 2$.

The inverse of a symmetric Matrix

The elementary transformations are to be transformed so that the property of being symmetric is preserved. This requires that the transformations occur in pairs, a row transformation must be followed immediately by the same column transformation.

Example-Find the inverse of the following matrix employing elementary transformations:

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution- The given matrix is $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$. As A=I$_3$A

So
$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to \frac{R_1}{3}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_2 \to -R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} A \quad R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_3 \to -3R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_1 \to R_1 - \frac{4}{3}R_3, \quad R_2 \to R_2 + \frac{4}{3}R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_1 \to R_1 + R_2$$

Hence, $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$. Ans.

Example-If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, Find two non-singular matrices P and Q such that $PAQ = I$.

Hence find A^{-1}.

Solution- Since $A_{3\times 3} = I_3 A I_3$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\begin{bmatrix}1 & 0 & 0\\0 & -3 & 4\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\-2 & 3 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} R_3 \to R_2 - 2R_1$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 3 & 4\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\-2 & 3 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} C_2 \to -C_2$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 3 & 4\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\-2 & 3 & 0\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\-2 & 3 & -3\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\-2 & 3 & -3\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & -1 & 1\\0 & 0 & 1\end{bmatrix} C_3 \to C_3 - C_2$$

$$I_3 = PAQ$$

$A^{-1} = QP,$

$$A^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\-2 & 3 & -3\end{bmatrix} \qquad \left[\begin{array}{l} I = PAQ \\ P^{-1} = AQ \\ P^{-1}Q^{-1} = A \\ (P^{-1}Q^{-1})^{-1} = A^{-1} \\ QP = A^{-1} \end{array}\right.$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix}1 & -1 & 0\\-2 & 3 & -4\\-2 & 3 & -3\end{bmatrix}$$ Ans.

Solution of Simultaneous Equations

Consider a system a linear equations as

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n = b_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots.a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

$$\Rightarrow \qquad AX = B$$

Here matrices A, X and B are called Coefficient, Variable and Constant matrices respectively.

If B=0 then system AX=0 is called Homogeneous and for B≠0, the system AX=B is called Non-Homogeneous.

Types of Linear Equations

(1) **Consistent.** A system of equations is said to be consistent, if they have one or more solution i.e.

$x+2y=4$	$x+2y=4$
$3x+2y=2$	$3x+6y=12$
Unique solution	Infinite solution

(2) **Inconsistent.** If a system of equation has no solution, it is said to be inconsistent i.e.

$$x+2y=4$$
$$3x+6y=5$$

Consistency of A System of Linear Equations

$$a_{11}\,x_1 + a_{12}\,x_2 + \ldots a_{1n}\,x_n = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + \ldots a_{2n}\,x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots a_{mn}\,x_n = b_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots.a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

$$\Rightarrow \qquad AX = B$$

and
$$[A,B] = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} & b_1 \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} & b_2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ a_{m1} & a_{m2}\ldots\ldots.a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix. $\boxed{[A:B]}$

The augmented matrix $[A : B]$ is reduced into Echelon form by elementary row transformations and find rank [A] and rank $[A : B]$. Now three cases arise:

(a) Consistent equations. If Rank [A] = Rank $[A : B]$

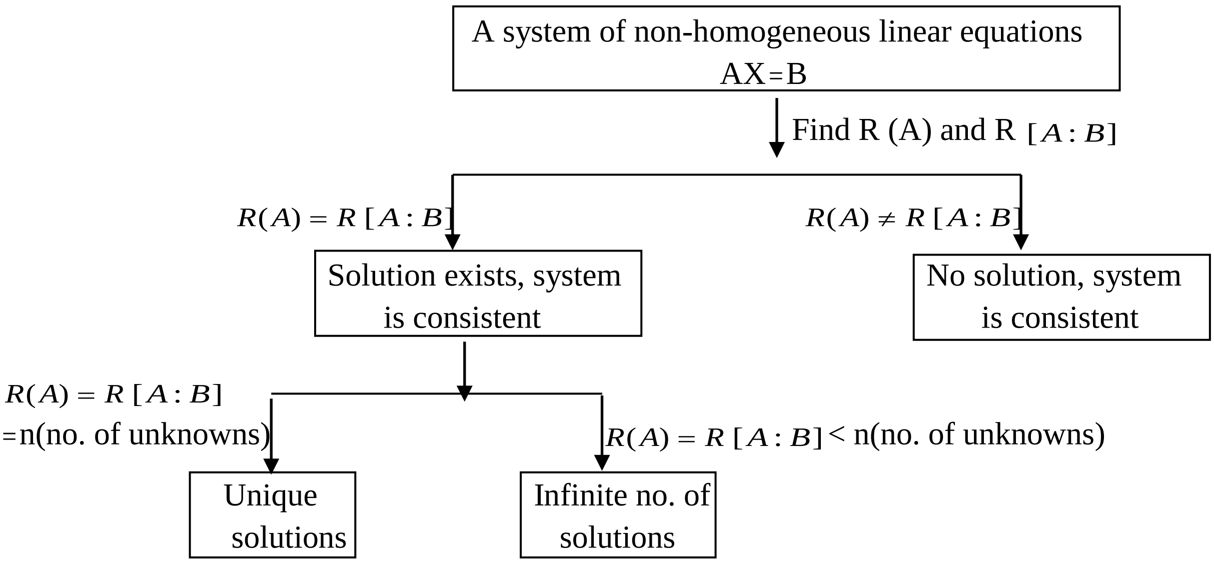
(i) Unique solution: Rank [A]= Rank $[A : B]$ =n (number of variables or unknowns)

(ii) Infinite solution: Rank [A] = Rank $[A : B]$ =r, r < n

(b) Inconsistent equations. If Rank [A] $\neq$ Rank $[A : B]$

At the end of the row transformation the value of z is calculated from the last equation and value of y and the value of x are calculated by the backward substitution.

In Brief :



Example-Test the consistency and hence solve the following set of equation.

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + x_2 - 2x_3 = 1$$
$$4x_1 - 3x_2 - x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 4$$

Solution- The given set of equations is written in the matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$AX = B$$

Here, we have augmented matrix $[A, B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3\,R_1 \\ R_3 \to R_3 - 4\,R_1 \\ R_4 \to R_4 - 2\,R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to -\frac{1}{5}\,R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 11\,R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \frac{1}{6}\,R_3$$

Number of non-zero rows $=$ Rank of matrix.

$$\Rightarrow \qquad R[A:B] = R(A) = 3$$

Hence, the given system is consistent and possesses and unique solution. In matrix form the system reduces to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 2$$
$$x_2 + x_3 = 1$$
$$x_3 = 1$$

From (2), $x_2 + 1 = 1 \Rightarrow x_2 = 0$

From (1) $x_1 + 0 + 1 = 2 \Rightarrow x_1 = 1$

Hence, $x_1 = 1,\ x_2 = 0$ and $x_3 = 1$

Example- Show that the equations

$$2x + 6y = -11,\ 6x + 20y - 6z = -3,\ 6y - 18z = -1$$

are not consistent.

Solution- Augmented matrix $[A : B]$

$$= \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix} R_3 \to R_3 - 3R_2$$

The rank $[A : B]$ of is 3 and the rank of A is 2

Rank of A $\neq$ Rank $[A : B]$.

The equations are not consistent.

Example- Investigate the values of λ and μ so that the equations:

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

Have (i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Solution- Here, we have,

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

The above equations are written in the matrix form

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & : & -\frac{47}{2} \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{7}{2}R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(i) No solution.

Rank $(A) \neq$ Rank $[A : B]$

$\lambda - 5 \neq 0$ or $\lambda = 5$ and $\mu - 9 \neq 0$ $\Rightarrow \mu \neq 9$

(ii) A unique solution

Rank $(A) = R[A : B] =$ Number of unknowns

$\lambda - 5 \neq 0$ $\Rightarrow$ $\lambda \neq 5$

(iii) An infinite number of solutions.

Rank $(A) =$ Rank $[A : B] = 2$.

$\lambda - 5 = 0$ and $\mu - 9 = 0$

$\lambda = 5$ and $\mu = 9$ Ans.

Homogeneous Equations

For a system of homogeneous linear equations $AX = O$

(i) $X = O$ is always a solution. This solution in which each unknown has the value zero is called the Null Solution or the trivial solution. Thus a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solutions.

(ii) If $R(A) =$ number of unknowns, the system has only the trivial solution.

(iii) If $R(A) <$ number of unknowns, the system has an infinite number of non-trivial solutions.

A system of homogeneous linear equations

AX=O

Always has a solution

$R(A) = R[A : B]$

$R(A)$ = n(no. of unknowns) $R(A)$ = n(no. of unknowns)

Unique of trivial Solution

Infinite no. of non-trivial solutions

(each unknown equal to zero)

Note.(i) For zero solution, $|A| \neq 0$.

(ii) For non-zero solutions, $|A| = 0$.

Example. Determine 'b' such that the system of homogeneous equations

$$2x + y + 2z = 0;$$
$$x + y + 3z = 0;$$
$$4x + 3y + bz = 0$$

has (i) Trivial solution

(ii) Non-Trivial solution. Find the Non-Trivial solution using matrix method.

Solution- Here, we have

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

(i) For trivial solution: We know that $x = 0,\ y = 0$ and $z = 0$. So b can have any value.

(ii) For non-trivial solution: The given equations are written in the matrix form as:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

$$R_1 \leftrightarrow R_2, \quad R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1, \quad R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & b-12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & b-8 \end{bmatrix}$$

For non-trivial solutions or infinite solutions $R(A) = 2 <$ Number of unknowns

$b - 8 = 0, \quad b = 8$ Ans.

Eigen values

Eigen values and eigen vectors are used in the study of ordinary differential equations, analysing population growth and finding powers of matrices.

Eigen Values

Let
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$AX = Y$$

Where A is the matrix of order nxn, X is the column vector and Y is also column vector of order nx1 ……(1)

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let X$\neq 0$ be a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. λX .

$$AX = Y = \lambda X$$
$$AX - \lambda IX = 0 \qquad \ldots(2)$$
$$(A - \lambda I)X = 0$$

Thus the unknown scalar λ is known as an eigen value of the matrix A and the corresponding non zero vector X as eigen vector.

The eigen values are also called characteristic values or proper values or latent values.

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$ Characteristic matrix

Characteristic Polynomial: The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Some Important Properties of Eigen Values

(1) Any square matrix A and its transpose A' have the same eigen values.

Note. The sum of the elements on the principle diagonal of a matrix is called the trace of the matrix.

(2) The sum of the eigen values of a matrix is equal to the trace of the matrix.

(3) The product of the eigen values of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(i) k A are $k\lambda_1,\ k\lambda_2,\ \ldots\ k\lambda_n$ (ii) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots.\lambda_n^m$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots., \frac{1}{\lambda_n}$.

Cayley-Hamilton Theorem

Satement. Every square matrix satisfies its own characteristic equation.

If $|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n) = 0$ be the characteristic equation of $n \times n$ matrix $A = [a_{ij}]$, then the matrix equation

$X^n + a_1X^{n-1} + a_2X^{n-2} + \ldots + a_nI = 0$ is satisfied by $X = A$ i.e.,

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = 0$$

Example-Verify Cayley- Hamilton theorem for the matrix

$$A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} \text{ and hence find } A^{-1}.$$

Solution- The characteristic equation of the matrix is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2\\ 2 & -1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(-1-\lambda)-4=0 \Rightarrow -1+\lambda^2-4=0 \Rightarrow \lambda^2-5=0$$

By Cayley-Hamilton Theorem,

$$A^2-5I=0 \qquad \ldots(1)$$

Now, $A^2=A.A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & -5\end{bmatrix}$

$$A^2-5I=\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}+\begin{bmatrix}-5 & 0\\ 0 & -5\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0 \qquad \ldots(2)$$

From (1) and (2), Cayley-Hamilton theorem is verified.
Again from (1), we have

$$A^2-5I=0$$

Multiplying by A^{-1}, we get

$$A-5A^{-1}=0 \Rightarrow A^{-1}=\frac{1}{5}A$$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & -\frac{1}{5}\end{bmatrix} \qquad \text{Ans}$$

Example-Find the characteristic equation of the matrix

$$A=\begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

Hence find A^{-1}.

Solution- Characteristic equation of matrix A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2 & -2\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{vmatrix}=0$$

$\Rightarrow$ $(1-\lambda)[(1-\lambda)(-1-\lambda)-3]-2(-1-\lambda-1)-2(3-1+\lambda)=0$

$\Rightarrow$ $(1-\lambda)(-1+\lambda^2-3)-2(-\lambda-2)-2(2+\lambda)=0$

$\Rightarrow$ $(1-\lambda)(\lambda^2-4)+2\lambda+4-4-2\lambda=0$

$\Rightarrow$ $(-\lambda+1)(\lambda^2-4)=0$ or $\lambda^3-\lambda^2-4\lambda+4=0$

By Cayley-Hamilton Theorem

$$A^3 - A^2 - 4A + 4I = 0$$

$\Rightarrow$ $$A^2 - A - 4I + 4A^{-1} = 0 \qquad \text{(Multiplying by } A^{-1}\text{)}$$

$\Rightarrow$ $$4A^{-1} = [-A^2 + A + 4I] \qquad \ldots(1)$$

Now $A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$

From (1), we have

$$4A^{-1} = -\begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-+1+4 & 2+2+0 & -2-2+0 \\ -3+1+0 & -6+1+4 & 2+1+0 \\ -3+1+0 & -2+3+0 & -2-1+4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

Characteristic vectors or Eigen vectors

As we have discussed in earlier a column vector X is transformed into column vector Y by means of a square matrix A.

Now we want to multiply the column vector X by a scalar quantity λ so that we can find the same transformed column vector Y.

i.e., $AX = \lambda X$

X is known as eigen vector.

<u>Properties of eigen vectors</u>

1. The eigen vector X of a matrix A is not unique.
2. If $\lambda_1, \lambda_2, ..., \lambda_n$ be distinct eigen values of an $n \times n$ matrix then corresponding eigen vectors $X_1, X_2, ..., X_n$ form a linearly independent set.
3. If two or more eigen values are equal it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots.
4. Two eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.
5. Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.

Example-Find the eigen value and corresponding eigen vectors of the matrix

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

Solution- $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda) - 4 = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda+1)(\lambda+6) = 0 \Rightarrow \lambda = -1, -6$$

The eigen values of the given matrix are -1 and -6.

(i) When $\lambda = -1$, the corresponding eigen vectors are given by

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$$

Let $x_1 = k$, then $x_2 = 2k$

Hence, eigen vector $X_1 = \begin{bmatrix} k \\ 2k \end{bmatrix}$

(i) When $\lambda = -6$, the corresponding eigen vectors are given by

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 = 0 \qquad \Rightarrow x_1 = -2x_2$$

Let $x_1 = k_1$, then $x_2 = -\frac{1}{2}k_1$

Hence eigen vector $X_2 = \begin{bmatrix} k_1 \\ -\frac{k_1}{2} \end{bmatrix}$ or $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$

Hence eigen vectors are $\begin{bmatrix} k \\ 2k \end{bmatrix}$ and $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$

Example-Show that the matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has less than three linearly independent eigen vectors. Is it possible to obtain a similarity transformation that will diagonalise this matrix?

Solution- $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0 \qquad \Rightarrow \qquad \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

By trial

Let $\lambda = 2$, then $8 - 28 + 32 - 12 = 0$. $\qquad \therefore\ (\lambda - 2)$ is one factor

2	1	−7	16	−12
		2	−10	12
	1	−5	6	0

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda 2-5\lambda+6)=0 \Rightarrow (\lambda-2)(\lambda-2)(\lambda-3)=0, \lambda=2,2,3$$

Eigen vector for $\lambda=3$

$$\begin{vmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{vmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 0x+10y+5z=0 \\ -2x-6y-4z=0 \end{matrix} \quad \Rightarrow \frac{x}{-40+30}=\frac{y}{-10-0}=\frac{z}{0+20} \Rightarrow \frac{x}{1}=\frac{y}{1}=\frac{z}{-2}$$

Eigen vector $=\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Eigen vector for $\lambda=2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{i.e.,} \begin{bmatrix} x+10y+5z=0 \\ 2x+5y+4z=0 \end{bmatrix} \Rightarrow \frac{x}{40-25}=\frac{y}{10-4}=\frac{z}{5-20}$$

$$\Rightarrow \quad \frac{x}{5}=\frac{y}{2}=\frac{z}{-5}=k, \text{ Eigen vector } =\begin{bmatrix} 5k \\ 2k \\ -5k \end{bmatrix}$$

We get one eigen vector corresponding to repeated root $\lambda_2=2=\lambda_3$.

Eigen vector corresponding to $\lambda_2=2=\lambda_3$ are not linearly independent.

Similarity transformation is not possible.

Example- Find the eigen values, eigen vectors the modal matrix and diagonalise the matrix given below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Solution-The characteristic equation of the given matrix is

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}=0$$

$$\Rightarrow \quad (1-\lambda)\{(3-\lambda)^2-1\}=0 \quad \Rightarrow \quad (1-\lambda)(3-\lambda+1)(3-\lambda-1)=0$$

$$\Rightarrow \quad (1-\lambda)(4-\lambda)(2-\lambda)=0 \quad \Rightarrow \quad \lambda=1,2,4$$

Eigen vectors

When $\lambda=1$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_3 \to R_3 + \frac{1}{2} R_2$$

$\Rightarrow \quad 2x_2 - x_3 = 0$ ...(1)

$\frac{3}{2} x_3 = 0 \Rightarrow x_3 = 0$ ...(2)

Putting $x_3 = 0$ from (2) in (1), we get $2x_2 - 0 = 0 \Rightarrow x_2 = 0$

Eigen Vector $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda = 2$,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_1 \to -R_1 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$x_1 = 0$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

Eigen vector $= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 4$,

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 = 0$

$-x_2 - x_3 = 0$

$x_2 = -x_3$

Eigen Vector $= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Model matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Let us diagonalise the given matrix:

$$P^{-1}AP = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & -4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Diagonalization of a matrix

Theorem. If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}\ AP$ is a diagonal matrix.

Example-Find a matrix P which diagonalizes the matrix

$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify $P^{-1}\ AP = D$ where D is the diagonal matrix.

Solution-The characteristic equation of matrix A is

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (4-\lambda)\ (3-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 12 - 2 = 0 \quad \Rightarrow \quad \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-5) = 0 \quad \Rightarrow \quad \lambda = 2, \lambda = 5$$

Eigen values are 2 and 5.

(i) When $\lambda = 2$, eigen vectors are given by the matrix equation

$$\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\quad 2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$

Let $\quad x_1 = k,\ x_2 = -2k$

Hence, the eigen vector $\quad X_1 = \begin{bmatrix} k \\ -2k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(ii) When $\lambda = 5$, eigen vectors are given by the matrix equation

$$\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\quad -x_1 + x_2 = 0 \Rightarrow x_1 = x_2$

Let $x_1 = k$, then $x_2 = k$

Hence, the eigen vector $\quad X_2 = \begin{bmatrix} k \\ k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Modal matrix $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \Rightarrow P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

For diagonalization

$$D = P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Verified.